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# ON REPRESENTATION OF INTEGERS BY INDEFINITE TERNARY QUADRATIC FORMS OF QUADRATFREI DETERMINANT

A DISSERTATION SUBMITTED TO THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1931

BY

ARNOLD E. ROSS

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# ON REPRESENTATION OF INTEGERS BY INDEFINITE TERNARY QUADRATIC FORMS OF QUADRATFREI DETERMINANT.*

By Arnold E. Ross.†

1. The object of this paper is a study of representation of integers by indefinite ternary quadratic forms whose determinant is free from square factors.

A. Meyer ‡ gave conditions under which an indefinite form $f$ with relatively prime invariants $\Omega$, $\Delta$ ($\Delta$ odd, $\Omega \not\equiv 0 \pmod 4$) represents an odd integer $m$ prime to $\Omega$ and not divisible by certain prime factors of $\Delta$. Dickson § extended these conditions to the case when only $m$ or both $m$ and $\Delta$ are double an odd integer, retaining all other restrictions of Meyer.

Employing a method of Dirichlet ¶ and Markoff's ‖ table of indefinite ternary forms Dickson ** studied representation of integers by forms of negative determinant $-D$, where $D \leqq 83$ and was either a prime, double a prime, the product of two distinct primes, or double such a product.

In this paper we generalize the above mentioned method of Dirichlet and employ this generalization and a theorem of Meyer †† to solve the problem of representation of integers by indefinite ternary quadratic forms whose determinant is odd and is free from square factors.

We shall prove the following

Theorem 1.‡‡ *Let $f$ be an indefinite ternary quadratic form whose*

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* Read before the American Mathematical Society, November, 1929.

† National Research Fellow.

‡ A. Meyer, *Vierteljahrsschrift Naturf. Gesellschaft Zürich*, Vol. 29 (1884), pp. 209-222.

§ L. E. Dickson, *Studies in the Theory of Numbers*, Ch. V, Chicago, 1930.

¶ G. L. Dirichlet, *Journal für Mathematik*, Vol. 40 (1850), pp. 228-232.

‖ A. W. Markoff, *Mém. Imp. Acad. Sc. St. Petersbourg*, series 8, Vol. 23 (1909), No. 7, 22 pp. For an extension of this table see L. E. Dickson, *Ibid.*, pp. 150-151.

** For the details of Professor Dickson's method see S. Silberfarb, "Representation by indefinite ternary quadratic forms," *Dissertation, University of Chicago*, 1929. Also R. H. Marquis, "The representation of integers," *Dissertation, University of Chicago*, 1929.

†† A. Meyer, *Journal für Mathematik*, Vol. 108 (1891), p. 139. See also L. E. Dickson, *Ibid.*, p. 54, Theorem 47.

‡‡ The last mentioned results of Dickson, Silberfarb, and Marquis are all instances of this theorem and its companion theorem for even determinants. For this latter see A. E. Ross, *Proceedings of the National Academy of Sciences*, Vol. 18 (1932), pp. 600-608, § 1.

*determinant is odd and free from square factors. Let* $\Omega$ *and* $\Delta$ *be the invariants of* $f$. *Then* $\Omega = \pm 1$. Write $\rho_1, \cdots, \rho_\alpha$ *for those odd prime divisors of* $\Delta$ *for which*

$$(F \mid \rho_i) = -(-\Omega \mid \rho_i) \qquad (i = 1, \cdots, \alpha), \tag{1.1}$$

*and let* $\pi_1, \cdots, \pi_\nu$ *be those odd prime divisors of* $\Delta$ *for which*

$$(F \mid \pi_j) = (-\Omega \mid \pi_j) \qquad (j = 1, \cdots, \nu). \tag{1.2}$$

*Then if* $\alpha$ *is even* $f$ *represents every integer* $a$ *of none of the types*

$$\rho_i^{2k+1}[n\rho_i + \mu(-\Delta_i, \rho_i)] \tag{1.3}$$

*and no integers of any of these types. Here* $n = 0, \pm 1, \pm 2, \cdots k = 0, 1, 2, \cdots, \Delta = \rho_i \Delta_i$ *and*

$$\begin{array}{l} \text{for a given integer } x \text{ prime to } p \\ \mu(x, p) \text{ runs over all the least residues} \\ \text{of } p \text{ satisfying } (\mu(x, p) \mid p) = (x \mid p). \end{array} \tag{1.4}$$

*If* $\alpha$ *is odd,* $f$ *represents every integer* $a$ *of none of the types* (1.3) *and*

$$4^k(8n - \Delta) \tag{1.5}$$

*where* $k = 0, 1, 2, \cdots$, *and no integers of any of these types.*

The above theorem shows that in the case of an indefinite ternary quadratic form $f$ whose determinant is odd and free from square factors, all integers not represented by $f$ form several families of arithmetical progressions depending in a simple way on the prime factors of the determinant and the generic characters of $f$. The same has been found to be true for indefinite ternary quadratic forms in a much more general * case than the one considered here.

It is of interest to note that in what follows our lemmas, including the extension of the Dirichlet method, apply to positive as well as to indefinite forms.

2. *Integers not represented by* $f$. In this section we study the relation between the generic characters of $f$ and integers not represented by $f$.

LEMMA 1. *Let* $f$ *be a primitive ternary quadratic (definite or indefinite) form with invariants* $\Omega$ *and* $\Delta$. *Let* $\delta$ *be a prime divisor of* $\Delta$ *for which*

$$(F \mid \delta) = -(-\Omega \mid \delta). \tag{2.11}$$

* These results will appear in a forthcoming paper. For a brief report see A. E. Ross, *ibid.*, §§ 4, 5, and 7.

*Further write* $\Delta = \delta\Delta_1$, *and let* $\delta$ *be prime to* $\Omega\Delta_1$. *Then* $f$ *does not represent integers of the form*

$$\delta^{2k+1}[n\delta + \mu(-\Delta_1, \delta)], \tag{2.12}$$

*where* $\mu(-\Delta_1, \delta)$ *runs over all those least residues of* $\delta$ *whose quadratic character is the same as that of* $-\Delta_1$, *i. e.*

$$(\mu(-\Delta_1, \delta) \mid \delta) = (-\Delta_1 \mid \delta). \tag{2.13}$$

Write

$$f = ax^2 + by^2 + cz^2 + 2ryz + 2sxz + 2txy, \tag{2.14}$$

and let

$$F = Ax^2 + By^2 + Cz^2 + 2Ryz + 2Sxz + 2Txy \tag{2.15}$$

be the reciprocal of $f$. Then *

$$aCf = CX^2 + \Omega Y^2 + \Omega\Delta az^2 \tag{2.16}$$

where

$$X = ax + ty + sz, \qquad Y = Cy - Rz. \tag{2.17}$$

We may assume † that $a$ and $C$ are relatively prime and have no odd prime factor in common with $\Omega\Delta$.

If $m$ is represented by $f$, then (2.16) holds with $f$ replaced by $m$. Let $m = m_1\delta$. Then

$$aCm_1\delta = CX^2 + \Omega Y^2 + \Omega\Delta az^2, \tag{2.18}$$

whence

$$CX^2 \equiv -\Omega Y^2 \pmod{\delta}$$

and in view of (2.11) and (2.15), $X \equiv Y \equiv 0 \pmod{\delta}$. Write $X = \delta X_1$, $Y = \delta Y_1$. Substituting into (2.18) and dividing through by the common factor $\delta$ we obtain $aCm_1 \equiv \Omega\Delta_1 az^2 \pmod{\delta}$, whence, since $a$ is prime to $\delta$,

$$Cm_1 \equiv \Omega\Delta_1 z^2 \pmod{\delta}. \tag{2.19}$$

If $m_1$ is prime to $\delta$, then $(m_1 \mid \delta) = (C\Omega\Delta_1 \mid \delta)$. Since by (2.11), $(C\Omega \mid \delta) = -(-1 \mid \delta)$, we have $(m_1 \mid \delta) = -(-\Delta_1 \mid \delta)$. Therefore if $(m_1 \mid \delta) = (-\Delta_1 \mid \delta)$, $m$ is not represented by $f$. This proves our lemma for the case of $k = 1$.

To complete the proof we note that if $f$ does not represent $l$, it does not

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* See A. E. Ross, *ibid.*, § 5.

† H. J. S. Smith, *Collected Mathematical Papers*, Vol. 1 (1894), §§ 5 and 9; L. E. Dickson, *Studies in the Theory of Numbers*, pp. 15-17, Chicago, 1930; P. Bachman, *Die Arithmetik der Quadratischen Formen v.* 1, p. 64.

represent $\delta^2 l$. For if we let $m = \delta^2 l$, then $m_1 \equiv 0 \pmod{\delta}$ and by (2.19), $\delta$ divides $z$. Thus $\delta$ divides all of $X$, $Y$, $z$ and hence by (2.17) also all of $x$, $y$, $z$, and therefore $f$ represents $l$, a contradiction.

LEMMA 2. *Let $f$ be a properly primitive ternary quadratic (definite or indefinite) form whose determinant is odd and is free from square factors. If (i) $f$ is definite and*

$$(F \mid \Delta) = (-\Omega \mid \Delta), \tag{2.21}$$

*or if (ii) $f$ is indefinite and*

$$(F \mid \Delta) = -(-\Omega \mid \Delta), \tag{2.22}$$

*then $f$ does not represent integers of the type*

$$4^k(8n - \Delta). \tag{2.23}$$

Since $f$ and $F$ are both properly primitive we may without loss of generality assume further that $a$ and $C$ are both odd.* We shall first prove our lemma in case $\Omega > 0$, whence $\Omega = 1$. Then $\Delta$ is positive or negative according as $f$ is positive or indefinite. Smith's character condition † in this case becomes

$$\Psi = (F \mid \Delta)(-1)^{\frac{1}{2}(\Delta+1)}$$

where

$$\Psi = (-1)^{\frac{1}{2}(\Omega C+1)\frac{1}{2}(\Delta a+1)} \tag{2.24}$$

Write $\Delta = e \mid \Delta \mid$. Then $\Psi = (F \mid \Delta)(-e)(-1 \mid \Delta)$. Therefore if $\Omega = 1$, in both cases (*i*) and (*ii*) of our lemma

$$\Psi = -1. \tag{2.25}$$

Relations (2.24) and (2.25) imply

$$\Omega C \equiv 1, \quad \Delta a \equiv 1 \pmod 4. \tag{2.26}$$

Hence, by (2.16),

$$aCf \equiv C(X^2 + Y^2 + z^2) \pmod 4. \tag{2.27}$$

If $f \equiv -\Delta_1 \pmod 8$, then $aCf \equiv -C \pmod 4$, whence $X^2 + Y^2 + z^2 \equiv 3 \pmod 4$, and therefore $X \equiv Y \equiv z \equiv 1 \pmod 2$. Then $-aC\Delta \equiv C + \Omega + \Omega\Delta a \pmod 8$. Hence

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* In fact to be able to do that we choose $C \equiv \Omega \pmod 4$ and hence the first one of congruences (2.26) will hold in any case. See L. E. Dickson, *Studies*, p. 15.

† H. J. S. Smith, *Collected Mathematical Papers*, Vol. 1 (1894), p. 470.

$$0 \equiv (a\Delta + 1)C + (a\Delta + 1)\Omega = (C + \Omega)(a\Delta + 1) \pmod 8.$$

But $a\Delta \equiv 1 \pmod 4$, and therefore $C \equiv -\Omega \pmod 4$ contrary to (2.261).

If $f = 4m \equiv 0 \pmod 4$, then $X^2 + Y^2 + z^2 \equiv 0 \pmod 4$ by (2.27). Therefore $X \equiv Y \equiv z \equiv 0 \pmod 2$, whence $x \equiv y \equiv z \equiv 0 \pmod 2$ by (2.17), and $f$ represents $m$. This proves Lemma 2 for $\Omega = 1$.

To complete the proof we need only note that every ternary quadratic form with $\Omega = -1$ is a negative * of one with $\Omega = 1$ and that if $f_1 = -f$, then $\Omega_1 F_1 = \Omega F$, $\Omega_1 C_1 = \Omega C$, and $\Delta_1 a_1 = \Delta a$.

3. *Dirichlet construction.* We consider a ternary quadratic (definite or indefinite) form $f$ whose determinant is odd and free from square factors. Let $a$ be an integer of none of the types not represented by $f$ in view of Lemmas 1 and 2. We shall in this section generalize a method of Dirichlet † to prove that every such integer $a$ is represented by some form $\phi$ in the same genus as $f$.

Adopting the already mentioned conventions for the sign of $\Omega$, we shall assume in what follows that $\Omega > 0$ which in our case implies that $\Omega = 1$. As we have pointed out in the closing paragraph of the preceding section this assumption does not essentially restrict the generality. Then as in the proof of Lemma 2, $\Delta$ is positive or negative according as $f$ is positive or indefinite.

If an integer $\gamma^2 a$ is not of the type (2.12) or (2.23) (or, what is the same, (1.3) or (1.5)), then $a$ is not of that type, and therefore we need to prove the desired result only for integers $a$ without square factors.

LEMMA 3. *Let $f$ be a ternary quadratic (positive or indefinite) form whose determinant is odd and free from square factors. Let $\Omega = 1$ and $\Delta$ be the invariants of $f$. Let $\rho_1, \cdots, \rho_\alpha$ be all of those positive odd prime divisors of $\Delta$ for which (1.1) holds and write $\pi_1, \cdots, \pi_\nu$ for those for which (1.2) holds.*

*(i) Let $f$ be positive. Then if $\alpha$ is odd and $a$ is positive, quadratfrei, and is of none of the types (1.3) or if $\alpha$ is even and $a$ is positive quadratfrei, and is of none of the types (1.3) and (1.5), there exists a positive form*

$$\phi = ax^2 + by^2 + cz^2 + 2ryz + 2sxz \tag{3.11}$$

*of the same genus as $f$, having $a$ as its leading coefficient and hence representing $a$ properly.*

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* For adopted conventions for the sign of $\Omega$, see H. J. S. Smith, *Collected Mathematical Papers*, Vol. 1 (1894), p. 456, or L. E. Dickson, *Studies*, p. 10.

† G. L. Dirichlet, *Journal für Mathematik*, Vol. 40 (1850), pp. 228-232; E. Laudau, *Vorlesungen über Zahlentheorie*, B. 1 (1927), pp. 123-125.

*(ii) Let $f$ be indefinite. Then if $\alpha$ is even and $a$ is quadratfrei and is of none of the types* (1.3) *or if $\alpha$ is odd and $a$ is quadratfrei and is of none of the types* (1.3) *and* (1.5), *there exists an indefinite form $\phi$ as in* (3.11), *of the same genus as $f$ and which has $a$ as the leading coefficient and hence represents $a$ properly.*

We wish to show thus that for every integer $a$ subject to the restrictions of our lemma we can choose integers $b$, $c$, $r$, $s$ so that the form (3.11) has for its invariants $\Omega = 1$ and $\Delta$, i. e.,

$$aA - bs^2 = \Delta, \quad \text{here} \quad A = bc - r^2, \tag{3.12}$$

and so that its generic characters coincide with those of $f$.

We shall proceed with the construction.

Write $\rho_1 \cdots \rho_a = R$, $\pi_1, \cdots, \pi_v = E$. Let $T = (a, R)$, $S = (a, E)$ and write

$$R = TP, \quad E = SQ, \quad a = TSa_1, \tag{3.13}$$

$$\Delta = eD \quad \text{where} \quad D = |\Delta| > 0. \tag{3.14}$$

Then

$$D = RE = PQST. \tag{3.15}$$

Since by the assumption $a$ is not of the form (1.3),

$$(a/\tau \mid \tau) = -(-\Delta/\tau \mid \tau) = -(-eD/\tau \mid \tau)$$

for every odd prime divisor $\tau$ of $T$. Hence by ($3.13_3$) and (3.15)

$$(a_1 \mid \tau) = -(-ePQ \mid \tau). \tag{3.16}$$

Write $p$, $q$ for odd prime divisors of $P$ and $Q$ respectively, and let $\pi$ be a power of 2. Take

$$s = T, \quad A = \pi Sd, \quad d = 8PQTu + v, \tag{3.17}$$

$$v \equiv S \pmod 8, \quad \pi Sa_1 v \equiv ePQ \pmod T, \tag{3.18}$$

$$(v \mid p) = -(-\pi S \mid p), \quad (v \mid q) = (-S\pi \mid q). \tag{3.19}$$

Write

$$\pi Sa_1 v - ePQ = T2^{\lambda}w, \tag{3.21}$$

where $w$ is odd. In view of (3.12) let

$$T^2 b = aA - eD. \tag{3.22}$$

Then by (3.13) and (3.17)

$$T^2b = T^2S[8\pi a_1PQSu + 2^\lambda w] = T^2S2^\lambda b_1,$$

where

$$2^\lambda b_1 = 8M + 2^\lambda w, \qquad M = \pi a_1 PQS. \tag{3.23}$$

By (3.21) every common divisor of $\pi Sa_1$ and $2^\lambda w$ divides also $PQ$, similarly every common divisor of $PQ$ and $w$ must divide $\pi Sa_1 v$. Since $\pi Sa_1 v$ is prime to $PQ$ each one of $\pi Sa_1$ and $PQ$ and hence their product $M$ is prime to $2^\lambda w$.

Next, by (3.21), (3.$13_3$), and (3.$18_1$),

$$2^\lambda w \equiv \pi aS - ePQT \pmod 8. \tag{3.24}$$

1°. If

$$\pi a_1 \equiv 0 \pmod 2, \tag{3.25}$$

then $\lambda = 0$ and $b_1$ in (3.23) is odd.

2°. Assume next that

$$\pi a_1 \equiv 1 \pmod 2, \quad \text{whence} \quad \pi = 1. \tag{3.26}$$

Then (3.24) becomes $2^\lambda w \equiv aS - ePQT \pmod 8$, and hence $2^\lambda w \equiv 0 \pmod 4$ if and only if $aS \equiv ePQT \pmod 4$, that is if and only if $a \equiv ePQST \pmod 4$. Therefore if

$$a \equiv -ePQST \equiv -\Delta \pmod 4, \tag{3.27}$$

then $\lambda = 1$ and $b_1$ in (3.23) is odd.

By (3.23) we may write

$$b_1 = \sigma Mu + w, \tag{3.28}$$

where $\sigma = 8$ or 4 according as $\lambda = 0$ or 1. Then for every $a_1$ and $\pi$ which satisfy (3.25) or (3.26) and (3.27), $b_1$ in (3.28) is an odd integer for every integral value of $u$. Moreover, the coefficients $\sigma M$ and $w$ of the arithmetical progression $\sigma Mu + w$ are relatively prime by the above. Hence by the Dirichlet * theorem on primes in an arithmetical progression, there are infinitely many primes of the form (3.28) and hence we may assume that $b_1$ is a positive odd prime not dividing $\Delta$. If also

$$(-A \mid b_1) = 1, \tag{3.29}$$

then there exists an integer $r$ such that $-A \equiv r^2 \pmod{b_1}$. Since $b_1$ is odd and prime to $\Delta$ we may choose $r$ odd and $\equiv 0 \pmod S$.

* G. L. Dirichlet, *Abhandlungen der Königlichen Akademie der Wissenschaften*, pp. 108-110, Berlin, 1837. E. Laudau, *Vorlesungen über Zahlentheorie*, B. 1 (1927), pp. 79-96.

Write $-A = r^2 - b_1c_1$. Then by $(3.17_2)$ and the choice of $r$, $c_1 \equiv 0 \pmod{S}$, and if $\pi = 1$, $c_1$ is even. We may write therefore $c_1 = 2^\lambda Sc$. Then

$$A = b_1 2^\lambda Sc - r^2 = bc - r^2.$$

The form $\phi$ thus determined has determinant $\Delta$. The adjoint $\Phi$ of $\phi$ is properly primitive since the g. c. d. of $A = bc - r^2$ and $B = ac - s^2$ is prime to $\Delta$ by (3.17) and $(3.18_2)$. Hence $|\Omega| = 1$. Since $b$ is positive, $\phi$ is positive if $a$ and $\Delta$ are positive, for then the two upper left hand corner principal minors $a$ and $ab$ of the determinant of $\phi$ and also the determinant itself are positive. If $\Delta$ is negative, then $\phi$ is indefinite since it represents a positive integer $b$ and has a negative determinant.* Hence, in accord with our conventions, $\Omega = 1$. The generic characters of $\phi$ with respect to the odd prime factors $\sigma$, $p$, $q$ of $S$, $P$, $Q$ respectively, are

$$(\Phi \mid \sigma) = (B \mid \sigma) = (ac - s^2 \mid \sigma) = (-T^2 \mid \sigma) = (-1 \mid \sigma),$$
$$(\Phi \mid p) = (A \mid p) = (\pi Sv \mid p) = -(-1 \mid p),$$
$$(\Phi \mid q) = (A \mid q) = (\pi Sv \mid q) = (-1 \mid q),$$

by $(3.13_3)$, (3.17), and (3.19). Since by (3.16) and $(3.18_2)$

$$(3.291) \qquad (v \mid \tau) = (\pi S a_1 v \mid \tau)(\pi S \mid \tau)(a_1 \mid \tau) = -(-\pi S \mid \tau),$$

the characters of $\phi$ with respect to every odd prime factor $\tau$ of $T$ are

$$(\Phi \mid \tau) = (A \mid \tau) = (\pi Sv \mid \tau) = -(-1 \mid \tau).$$

Hence if (3.29) holds, the constructed form $\phi$ is the one desired in Lemma 3.

To complete the proof of our lemma it remains thus to show that for every integer $a$ satisfying conditions of our lemma we can choose $\pi$ so that either (3.25) or (3.26) and (3.27) hold and so that (3.29) is true.

For every $a_1$ and $\pi$ satisfying (3.25) or (3.26) and (3.27), and for every integral value of $u$ we have, in view of (3.17), (3.18), and (3.22),

$$(-A \mid b_1) = (-\pi \mid b_1)(Sd \mid b_1)$$
$$(Sd \mid b_1) = (b_1 \mid Sd) = (T^2 2^\lambda b_1 \mid Sd) = (-ePQT \mid Sd)$$
$$= (PQT \mid Sd) = (Sd \mid PQT) = (Sv \mid PQT).$$

By (3.19) and (3.291)

$$(v \mid PQT) = (-1)^a(-\pi S \mid PQT).$$

Therefore

$$(3.31) \qquad (-A \mid b_1) = (-1)^a(-1 \mid b_1)(-1 \mid PQT)(\pi \mid b_1)(\pi \mid PQT).$$

* See L. E. Dickson, *Studies*, p. 10, § 7. One can verify these assertions directly by multiplying (2.16) by $\Omega$ and replacing $f$ by $\phi$ and $\Omega C$ by $ab$.

Let, $(i_1)$, $f$ be positive and $\alpha$ be odd or, $(ii_1)$, $f$ be indefinite and $\alpha$ be even. Then for every * $a$ not of the types (1. 3) we take $\pi = 4$. Then

$$b_1 \equiv w \equiv -ePQT \pmod 4$$

by (3. 28) and (3. 21). Since $e = +1$ or $-1$ according as $f$ is positive or indefinite, (3. 29) holds in both of the above cases by (3. 31).

Next, let $(i_2)$, $f$ be positive and $\alpha$ be even or, $(ii_2)$, $f$ be indefinite and $\alpha$ be odd. Let $a$ be an integer of none of the types (1. 3) and (1. 5).

If

$$a \not\equiv -\Delta \equiv -ePQST \pmod 4 \tag{3. 32}$$

we take $\pi = 2$. Then

$$b_1 \equiv w \pmod 8 \tag{3. 33}$$

by (3. 28). If $a$ is even, whence $a$ is double an odd,

$$w \equiv -ePQT + 4 \pmod 8 \tag{3. 34}$$

by (3. 21). Remembering again that $e = +1$ or $-1$ according as $f$ is positive or indefinite, we see that (3. 29) holds by (3. 31). Next let $a$ be odd. Then $a \equiv ePQST \pmod 4$ by (3. 32) and

$$w \equiv -ePQT + 2ePQT \equiv ePQT \pmod 8.$$

Then $b_1 \equiv ePQT$ and (3. 31) implies (3. 29).

If $a \equiv -\Delta \pmod 4$, then

$$a \equiv -ePQST + 4 \pmod 8$$

since $a$ is not of the form (1. 5). In this case we take $\pi = 1$. Then $b_1 \equiv w \pmod 4$. Also, by (3. 21) and (3. 35), $2w \equiv -2ePQT + 4 \pmod 8$, whence $w \equiv -ePQT + 2 \pmod 4$. Then

$$b_1 \equiv -ePQT + 2 \pmod 4$$

and again (3. 31) implies (3. 29).

4. *Proof of Theorem* 1. Lemmas 1 and 2 show that no integer of the form (1. 3) or (1. 3) and (1. 5) according as $\alpha$ is even or odd, are represented by an indefinite form $f$ satisfying the conditions of Theorem 1. If $\Omega = 1$, Lemma 3 shows that every integer not excluded by these lemmas is represented by some form $\phi$ in the genus of $f$. But by Meyer's † theorem

* If $f$ is positive we need only to consider positive integers $a$.

† A. Meyer, *Journal für Mathematik*, Vol. 108 (1891), p. 139; L. E. Dickson, *Studies*, p. 54.

there is but one class in every genus of indefinite ternary quadratic forms of determinant wihch is free from square factors. Hence $\phi$ is equivalent to $f$, and $a$ is represented by $f$ as well as by $\phi$. If $\Omega = -1$ we augment Lemma 3 by an argument similar to that at the end of Section 2.

In case $f$ is a positive ternary form of odd quadratfrei determinant, and if the genus of $f$ contains but one class,* Lemmas 1, 2, and 3 permit us to write down at once the families of the arithmetical progressions giving the totality of integers not represented by $f$.

For example consider forms

$$f_1 = x^2 + 2y^2 + 3z^2 - 2yz$$

and

$$f_2 = 2x^2 + 2y^2 + 3z^2 + 2yz + 2xz + 2xy.$$

Determinants of $f_1$ and $f_2$ are equal to 5 and 7 respectively. Their generic characters are

$$(F_1 \mid 5) = (2 \mid 5) = -1 = -(-\Omega \mid 5) = (F_1 \mid \Delta),$$
$$(F_2 \mid 7) = (3 \mid 7) = -1 = (-\Omega \mid 7) = (F_1 \mid \Delta).$$

Their genera contain but one class each. (See L. E. Dickson, *Studies in the Theory of Numbers*, p. 181, Chicago, 1930.) Applying Lemmas 1 and 3 to $f_1$ we see that $f_1$ represents all integers save those of the form $5^{2k+1}(5n+1)$ or $5^{2k+1}(5n+4)$. Similarly Lemmas 2 and 3 show that $f_2$ represents all integers not of the form $4^k(8n+1)$.

California Institute of Technology.

* Representation of integers by a genus of positive ternary forms has been studied by Jones (see B. W. Jones, *Transactions of the American Mathematical Society*, Vol. 33 (1931), pp. 92-110, 111-124). However his final results involve the auxiliary parameters $\alpha$, $\beta$, $\gamma$ of a lemma of Smith (H. J. S. Smith, *Collected Mathematical Papers*, Vol. 1 (1894), p. 460).